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Causal Statistical Mechanics Calculation of Initial Cosmic Entropy and Quantum Gravity Prospects

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We report on the consequences of applying the causal laws to the statistical mechanical calculation of the initial cosmic entropy. It can be deduced that the entropy density tends to zero as one approaches the initial singularity. This result along with some quantum gravity considerations lead us to the conclusion that the initial data of the physical fields are asymptotically homogeneous in that limit.

1. INTRODUCTION

The observation of the cosmic microwave background is probably the most influential one in cosmology. In particular, the $\Delta T/T < 10^{-5}$ measurements [1] of the large-scale cosmic microwave background anisotropies establish an impressive isotropic behavior, which in turn gives us information on the initial cosmic data.

The existence of the cosmic microwave background was predicted before it first detection from the observation that in the past the universe has been always expanding. But when one puts this picture in the framework of general relativity, another striking prediction arises, namely the existence of an initial cosmic singularity. Through the singularity theorems [2] one can prove that under very general circumstances, the initial singularity is unavoidable.

There are many fundamental consequences associated with the existence of the initial cosmic singularity, in particular those related to the causality principle.

Many of the calculations done in cosmology are carried out using the so-called standard cosmological model, in which it is assumed that the Universe is

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isotropic and homogeneous. At very early cosmic times there is an era dominated by radiation and its geometry is given by the corresponding Friedmann line element, which is a special case of the Robertson–Walker metrics. One of the consequences of the existence of the cosmic singularity and the validity of the causality principle in the context of the standard cosmological model is that, let us say, opposite directions in the sky receive information on the cosmic microwave background radiation coming from regions that were causally disconnected at the time of emission. This is due to the fact that any region of the spacetime can only receive signals from other regions in its causal past, which because of the initial cosmic singularity is limited by the so-called particle horizons [3].

Then it is deduced that the observed homogeneity of the universe at very early epochs cannot be explained in terms of the thermodynamic equilibrium of regions that were causally disconnected.

In this article we study the implications that arise in the calculation of entropy from statistical mechanics in a universe that has been expanding, but when no isotropy or homogeneity is assumed. Even when one relaxes the hypothesis of isotropy and homogeneity, the singularity theorems predict in an ever-expanding universe the existence of a past singularity, so our next calculations will take this into account.

In order to give some perspective to the calculations, in the next section we review the thermodynamic derivation of the entropy of radiation from the knowledge of the equations of state. In Section 3 the entropy of radiation is calculated from causal statistical mechanics. Some remarks concerning quantum gravity considerations are presented in Section 4. Finally, Section 5 is devoted to brief comments.

2. ENTROPY OF RADIATION FROM THERMODYNAMICS

In order to simplify the discussion we will consider the contribution to the entropy coming from the electromagnetic radiation, which at very early cosmic times is actually the main contribution [4] to the energy density.

From a thermodynamics point of view, the entropy can be calculated if one knows the necessary equations of state, which for radiation are

$$U = aVT^4$$

and

$$P = \frac{1}{3}\frac{U}{V}$$

where U, V, P, and T are the internal energy, volume, pressure, and temperature, respectively, and a is a constant which is related to the Stefan–Boltzmann constant σ and the velocity of light c by the relation $a = 4\sigma/c$.

One can then prove from thermodynamic arguments [5] that, if these equations of state hold, the entropy is given by

$$S = \frac{4}{3} a^{1/4} U^{3/4} V^{1/4} \tag{1}$$

3. ENTROPY FROM CAUSAL STATISTICAL MECHANICS

The most detailed information that we have on the issue of the cosmic initial data comes from the observation of the temperature of the cosmic background radiation; then one can say that in the early universe, matter was in a state that allowed a local description in terms of thermodynamic quantities. Accordingly, we will concentrate on the role of the entropy.

Although in the early universe the local physical properties were constantly changing in time, one can think of taking a volume small enough so that the thermodynamic characteristic times are much smaller than any other dynamical time associated with the small subsystem. So, given a comoving observer with the matter flow, with world line $\gamma(\tau)$ and a small volume V_0, there is associated the entropy

$$S_\tau = S_\tau(U, V_0, N_X)$$

at proper time τ, where N_X represents all the other possible extensive quantities.

For the sake of simplicity we will only refer in the next calculation to the contribution to the entropy coming from radiation.

In statistical mechanics, one bridges the gap between the microscopic description of matter and its macroscopic thermodynamic description by the expression

$$S = -k_B \operatorname{tr}(\rho \ln \rho) = -k_B \sum_i P_i \ln P_i \tag{2}$$

where the first is the standard quantum mechanical expression, while the second refers to a particular diagonalizing basis and k_B is the Boltzmann constant.

The distribution probability function P_i must be calculated from the information on the thermodynamic boundary conditions. Since the chemical potential of the photon gas is zero, one can equivalently use the canonical or grand canonical formalism to treat this case. The distribution is then given by

$$P_i = \frac{1}{Z(\beta, V_0)} e^{-\beta E_i(V_0)}$$

where

$$Z(\beta, V_0) \equiv \sum_i e^{-\beta E_i}$$

from which the entropy becomes

$$S = k_B(\ln Z + \beta U)$$

Consistency with thermodynamics requires $\beta = 1/(k_B T)$.

In order to calculate the partition function Z we must describe more in detail what is meant by the index i in the distribution. Let $n_{k,\epsilon}$ be the number of photons with momentum k and polarization ϵ; then for a particular value of the energy in the state i we mean

$$E_i \to E\{n_{k,\epsilon}\} = \sum_{k,\epsilon} \hbar\omega(k) n_{k,\epsilon}$$

with $\omega(k) = c|k|$, and $n_{k,\epsilon}$ being nonnegative integers. Then the partition function is expressed [6] by

$$Z = \sum_{n_{k,\epsilon}} \exp\left[-\beta \sum_{k,\epsilon} \hbar\omega(k) n_{k,\epsilon} \right] = \prod_{k,\epsilon} \sum_n e^{-\beta\hbar\omega(k)n} = \prod_{k,\epsilon} \frac{1}{1 - e^{-\beta\hbar\omega(k)}}$$

from which we obtain

$$\ln Z = -\sum_{k,\epsilon} \ln(1 - e^{-\beta\hbar\omega(k)})$$

The internal energy U can be calculated from the expression

$$U = -\frac{\partial \ln Z}{\partial\beta} = \sum_{k,\epsilon} \frac{\hbar\omega(k) e^{-\beta\hbar\omega(k)}}{(1 - e^{-\beta\hbar\omega(k)})}$$

The entropy is then given by

$$S = k_B\left(-\sum_{k,\epsilon} \ln(1 - e^{-\beta\hbar\omega(k)}) + \beta \sum_{k,\epsilon} \frac{\hbar\omega(k)}{e^{\beta\hbar\omega(k)} - 1} \right)$$

As usual it is convenient to proceed with the calculation in the continuum representation, so that the entropy is expressed in terms of integrals, that is,

$$S = k_B\left(\int \left[-\ln(1 - e^{-\beta\hbar\omega}) + \beta \frac{\hbar\omega}{(e^{\beta\hbar\omega} - 1)} \right] \frac{V_0}{\pi^2 c^3} \omega^2 \, d\omega \right) \tag{3}$$

and introducing the variable of integration $w = \beta\hbar\omega$, one obtains

$$S = \frac{V_0 k_B}{\pi^2 c^3 \beta^3 \hbar^3}\left(\int [-\ln(1 - e^{-w})]w^2\, dw + \int \frac{w^3}{e^w - 1}\, dw\right) \tag{4}$$

Up to now we have just reproduced the calculation of the entropy of radiation as it appears in textbooks, but we have omitted on purpose the limits of the integrals; it is exactly here that causality plays a decisive role.

Let us recall that all these calculations are done in a small volume V_0 when the observer is at the point $\gamma(\tau)$ of the worldline γ. The standard limits of integration for expression (3) are from 0 to infinity; however, these standard limits do not take into account the existence of the initial cosmic singularity and the causality principle.

Due to the fact that there exists a particle horizon, the subsystem under consideration does not have available an infinite reservoir, but rather, because of causality, there is available only a finite amount of all the extensive variables. To fix ideas let us say that at the point $\gamma(\tau)$ the observer detects in his past for the extensive quantities X_α the total amount $X_{\alpha\max}(\tau)$. Then for the statistical description of our subsystem, one can think that it belongs to an ensemble of systems which in turn form a closed system with extensive quantities $X_{\alpha\max}(\tau)$. That is, one can think of one of the standard derivations of the properties of the canonical ensemble in terms of the study of a small subsystem of a big system with the microcanonical distribution [7].

The real situation then is that our subsystem, with the small volume V_0, does not have at its disposal an infinity of energy states. All this means that $\hbar\omega$ cannot run in the process of integration up to infinity, but to a certain maximum value that we call $U_{\max}$, and which is identified with the maximum available total energy to the subsystem, because of the existence of a particle horizon. Therefore the expression for the entropy is actually

$$S(\tau) = \frac{V_0 k_B}{\pi^2 c^3 \beta(\tau)^3 \hbar^3}\left(\int_0^{x_{\max}(\tau)} [-\ln(1 - e^{-w})]w^2\, dw + \int_0^{x_{\max}(\tau)} \frac{w^3}{e^w - 1}\, dw\right)$$

where $x_{\max}(\tau) = \beta U_{\max}(\tau)$. By an integration by parts of the first term, one obtains

$$S = \frac{V_0 k_B}{\pi^2 c^3 \beta^3 \hbar^3}\left(-\ln(1 = e^{-w})\left.\frac{w^3}{3}\right|_0^{x_{\max}} + \frac{4}{3}\int_0^{x_{\max}} \frac{w^3}{e^w - 1}\, dw\right) \tag{5}$$

When $x_{\max} = \infty$ the first term does not contribute to the value of the entropy, and one reproduces the result of equation (1); however, this is not the case in our calculation.

We are interested in the regime for $\tau \to 0$, which can be arranged to coincide with the limit when one approaches the initial cosmic singularity along the worldline $\gamma(\tau)$. It is clear that as τ decreases, so does $U_{\max}(\tau)$, and that in the limit for $\tau \to 0$, one has

$$\lim_{\tau \to 0} U_{\max}(\tau) = 0 \tag{6}$$

since the causal past disappears in this regime. This is not in contradiction with the fact that the energy density is expected to blow up when $\tau \to 0$.

It is also the case that if an observer along the worldline $\gamma(\tau)$ sees an expanding universe, like the one in which we live, then the temperature must increase for decreasing values of τ, or equivalently $\beta(\tau)$ must decrease for descending values of τ.

In summary $x_{\max}(\tau) \to 0$ when $\tau \to 0$, and we can make an asymptotic expansion of the entropy for very small values of τ. The first-order contribution of the two leading terms is

$$S \cong \frac{V_0 k_B}{\pi^2 c^3 \beta^3 \hbar^3} \left(-\frac{x_{\max}^3}{3} \ln x_{\max} + \frac{4}{9} x_{\max}^3 \right) \tag{7}$$

Recalling the definition of $x_{\max}$, the last expression reduces to

$$S(\tau) \cong \frac{V_0 k_B}{\pi^2 c^3 \hbar^3} \left(-\frac{U_{\max}(\tau)^3}{3} \ln(\beta(\tau) U_{\max}(\tau)) + \frac{4}{9} U_{\max}(\tau)^3 \right) \tag{8}$$

where now the factor multiplying the parenthesis is a constant independent of τ.

It is observed from equation (8) that although the last term goes to zero as $\tau \to 0$, it is not clear what happens with the first term due to the appearance of $\beta(\tau)$ in the logarithm. In order to study the behavior of the first term it is useful to see how the internal energy behaves in this regime.

The energy is given by

$$U = \frac{V_0}{\pi^2 c^3 \beta^4 \hbar^3} \int_0^{x_{\max}} \frac{w^3}{e^w - 1} \, dw \tag{9}$$

and its leading-order behavior for $\tau \to 0$ is given by

$$U(\tau) \cong \frac{V_0}{\pi^2 c^3 \beta(\tau)^4 \hbar^3} \frac{x_{\max}(\tau)^3}{3} = \frac{V_0}{\pi^2 c^3 \beta(\tau) \hbar^3} \frac{U_{\max}(\tau)^3}{3} \tag{10}$$

It is clear that the internal energy $U(\tau)$ of our subsystem cannot have more energy than the total observed energy $U_{\max}(\tau)$; therefore one has

$$\frac{V_0}{\pi^2 c^3 \beta(\tau) \hbar^3} \frac{U_{\max}(\tau)^3}{3} < U_{\max}(\tau) \tag{11}$$

in this regime. From this last inequality it is deduced that

$$\ln(\beta(\tau)U_{\max}(\tau)) > \ln\left(\frac{V_0}{\pi^2 c^3 \hbar^3}\frac{U_{\max}(\tau)^3}{3}\right) \tag{12}$$

which implies

$$\lim_{\tau\to 0}\left(-\frac{V_0}{\pi^2 c^3 \hbar^3}\frac{U_{\max}(\tau)^3}{3}\ln(\beta(\tau)U_{\max}(\tau))\right) = 0 \tag{13}$$

It is concluded then that the entropy of our subsystem vanishes in this limit, that is,

$$\lim_{\tau\to 0} S(\tau) = 0 \tag{14}$$

Although we have considered up to now only the contribution of the entropy coming from radiation, the calculations can be extended to include the other types of particles and then we will conclude that the total entropy $S_{tot}(\tau)$ associated with the small volume V_0 also goes to zero as $\tau \to 0$.

4. QUANTUM GRAVITY PROSPECTS

It has sometimes been stated that probably the smooth spacetime description for physical phenomena might be a successful picture only for relatively large scales, but it is conceivable that at very small scales one might need another kind of theory. The natural small-distance scale is given by the Planck scale, which is defined in terms of the Planck constant, the gravitational constant, and the velocity of light. It is possible that the yet-to-be-constructed theory of quantum gravity might show (and might need) a new structure of the spacetime at such small scales.

It is commonly believed that for low-energy processes and for systems where the gravitational fields are relatively weak, one need not be troubled with this possibly discrete nature of physical phenomena. However, it is not at all clear that one should avoid this issue when one is studying the physics in the vicinity of the initial cosmic singularity since in this case one has both high-energy processes and strong gravitational fields; therefore the implications of a different quantum gravity nature of the spacetime might be noticeable.

If the fundamental structure of the spacetime is actually of a discrete kind, then the description of physical fields at very small scales would be in terms of a finite set of variables for a given finite portion of the spacetime. In other words, physical fields will be described by a finite set of degrees of freedom.

Let us here adopt the attitude that the successful present description of physical phenomena in terms of a smooth spacetime is actually a sort of coarse-grained view of an underlying discrete structure, and therefore associated with any finite portion of the smooth (coarse-grained view of the) spacetime there is actually only a finite set of degrees of freedom. Furthermore, let us assume that each of these degrees of freedom is describable in terms of a finite set of integers. It is appropriate to emphasize that this point of view is in complete agreement with observations since in particular any physical measurement has as an output a finite set of numbers along with their respective experimental errors; in other words, it is a fact of nature that we cannot measure an infinity of degrees of freedom in any observation. However, the description of the physical systems fundamentally changes if one adopts this viewpoint.

The singularity theorems in an expanding universe predict the existence of incomplete past-directed timelike geodesics. How is one supposed to understand the prediction of the existence of the initial singularity, based on a smooth description of the spacetime, when a fine-grained discrete structure is assumed? There are clearly two possibilities; either the extensions of the past-directed timelike geodesics of the fine-grained view of the spacetime are still incomplete or not. In other words, either there remains an initial time even in the discrete description of the spacetime or the discrete manifold extends indefinitely to its past. The physics community seems to be divided in this respect; some researchers expect that a complete theory of quantum gravity will wash out an unpleasant cosmic singularity, while others, who are not so disturbed with an initial singularity, anticipate that some kind of singularity will remain even in a discrete description of the spacetime. We adopt here the second attitude; that is, that there will be a beginning of time even if one has a complete fine-grained description of the spacetime.

Given a region U of the spacetime, then the characterization of the physical fields will be given by a finite set of integers. To this set one can associate a measure. Let $I(U)$ be a measure of the information needed to describe the physical fields in the open 4-dimensional region U.

We only require the measure of information to have the following two elementary properties:

(A) Nonnegativity: the measure of information $I(S)$ of a system S is nonnegative,

$$I(S) \geq 0$$

(B) Subadditivity: the measure of information $I(U)$ of a system U, which is the disjoint union of two sets, that is, $U = S_1 \cup S_2$, is bounded by the sum of the information $I(S_1)$ and $I(S_2)$, that is,

$$I(U) \leq I(S_1) + I(S_2)$$

Given this measure and any point x of the spacetime, one can define the notion of information 4-density scalar, with respect to the 4-volume element ϵ, from the relation

$$I_{(4)}(x) = \lim_{U \to \phi; V_U \to 0} \frac{I(U)}{V_U}$$

with $V_U \equiv \int_U \epsilon$ being the volumes of the sequence of 4-dimensional open neighborhoods U which contain the point x. The regions U are thought of in the form $I^-(x_F) \cap I^+(x_P)$, that is, the intersection of the chronological past of x_F with the chronological future of x_P, where $x_F \in I^+(x)$ and $x_p \in I^-(x)$; and so the limit is obtained when both x_F and x_P approach x. It is deduced from property (B) that $I_{(4)}(x)$ is bounded; that is, the information 4-density scalar cannot be given in terms of distributions.

When one applies the causal law to the physical fields evaluated in the region U, one concludes that the information available at U can only depend on the information contained in its causal past.

Let x_τ be the point $\gamma(\tau)$ in the timelike curve γ, and let U_τ be an open neighborhood of x_τ. Then, if $J^-(U_\tau)$ denotes the causal past of U_τ, it is deduced from the causal law that

$$I(U_\tau) \leq I(J^-(U_\tau) - U_\tau) \leq \int_{J-(U_\tau)-U_\tau} I_4 \epsilon$$

where, while the first inequality is explained by causality, the second is deduced from the properties (A) and (B) of the measure.

Let the curve $\gamma(\tau)$ be a geodesic of maximal length from the point x reaching the initial singularity at $\tau = 0$. Then when we take the limit for the point x_τ to approach the initial singularity, we observe that the region $J^-(U_\tau) - U_\tau$ gets smaller and smaller, from which we deduce that actually

$$\lim_{\tau \to 0} I(U_\tau) = 0$$

since no causal past is left in this limit.

5. FINAL COMMENTS

The behavior of entropy as one approaches the initial singularity has been the subject of discussion from different perspectives. In general one finds efforts to explain a desired low-entropy behavior [8, 9]. The results of Section 3 show that the explanation of the low-entropy behavior comes just from the calculation in the framework of causal statistical thermodynamics. Our results should be contrasted with the standard treatment of entropy in the Friedmann cosmological model, where, if we call $S_\tau^{\rm std}$ the result of the standard calculation, one has that $S_\tau^{\rm std}$ of a comoving volume is constant, and therefore $S_\tau^{\rm std}(V_0)$ diverges in the limit $\tau \to 0$ for a small constant proper volume V_0, as is calculated for a comoving cosmic observer.

Recalling from equation (2) that $S = -k_B \, \Sigma_i P_i \ln P_i$, we conclude that in the asymptotic regime for τ going to zero, the physical fields will only have available a single limiting state.

The same situation is encountered from the calculations of the last section. Therefore, both results independently imply that a physical field in the limit along the curve $\gamma_p(\tau)$ for $\tau \to 0$ will only have available a single limiting state. But the same applies to any other curve $\gamma_q(\tau)$. One concludes then that if p and q are two distinct boundary spacetime points at the singularity, an observer at x sees the same behavior along the direction toward p and the direction toward q. Then as x approaches the initial singularity along the timelike curve $\gamma(\tau)$, one observes an asymptotic isotropic universe.

Since this is a property ascribed to any worldline $\gamma(\tau)$ which reaches the initial cosmic singularity at different spacetime boundary points, it is concluded that the limiting available data for the spacelike initial cosmic singularity must be homogeneous.

In conclusion, the observed initial isotropic and homogeneous behavior of the universe is not in contradiction with the existence of the initial singularity; on the contrary, it is a consequence of causality and the existence of the initial cosmic singularity.

Due to limitations of space we have left some technical points aside. A detailed study of the calculations and implications of the results reported in the previous sections will be done in a separate work.

REFERENCES

[1] J. C. Mather, C. L. Bennett, M. W. Boggess, M. G. Hauser, G. F. Smoot, and E. L. Wright, Recent results from COBE, in *General Relativity and Gravitation 1992*, R. J. Gleiser, C. N. Kozameh, and O. M. Moreschi, eds., IOP, Bristol (1993).

[2] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time*, Cambridge University Press, Cambridge (1973).

[3] W. Rindler, Visual horizons in world-models, *Mon. Not. R. Ast. Soc.* **116**, 663 (1956).

[4] P. J. E. Peebles, *Physical Cosmology*, Princeton University Press, Princeton, New Jersey (1971).

[5] H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics*, 2nd ed. Wiley, New York (1985).

[6] K. Huang, *Statistical Mechanics*, 2nd ed. Wiley, New York (1987).

[7] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Pergamon Press, Oxford (1980).

[8] R. Penrose, Gravity and quantum mechanics, in *General Relativity and Gravitation 1992*, R. J. Gleiser, C. N. Kozameh, and O. M. Moreschi, eds. IOP, Bristol (1993).
[9] J. J. Halliwell, The interpretation of quantum cosmological models, in *General Relativity and Gravitation 1992*, R. J. Gleiser, C. N. Kozameh, and O. M. Moreschi, eds. IOP, Bristol (1993).
[10] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley, New York (1972).